Diamond (on the regulars) can fail at any strongly unfoldable cardinal

Mirna Džamonja
University of East Anglia
Norwich, NR4 7TJ, UK
http://www.mth.uea.ac.uk/people/md.html

Joel David Hamkins
The City University of New York
Mathematics, 365 Fifth Avenue, New York, NY 10016
http://jdh.hamkins.org

February 1, 2008

Abstract. If κ is any strongly unfoldable cardinal, then this is preserved in a forcing extension in which $\diamondsuit_\kappa(\text{REG})$ fails. This result continues the progression of the corresponding results for weakly compact cardinals, due to Woodin, and for indescribable cardinals, due to Hauser.

1 Introduction

For any stationary subset E of a regular uncountable cardinal κ the combinatorial diamond principle $\diamondsuit_\kappa(E)$ asserts that there is a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$

MSC 03E55, 03E35, 03E05. Keywords: unfoldable cardinals, infinite combinatorics, diamond sequence, Laver function. The research of the first author has been supported by an EPSRC Advanced Fellowship. The research of the second author has been supported by grants from Georgia State University, The Research Foundation of The City University of New York and the National Science Foundation. He is affiliated with the College of Staten Island of CUNY and the CUNY Graduate Center.

such that for any $A \subseteq \kappa$ the set $\{\, \alpha \in E \mid A \cap \alpha = A_\alpha \,\}$ is stationary in κ. It is relatively easy to show, for example, that if κ is a measurable cardinal, then $\diamondsuit_\kappa(\text{REG})$ holds, where REG is the set of regular cardinals (see Observation 1). The same is true if κ is merely ineffable or even subtle, by results of Jensen and Kunen (see [Kan88]). Furthermore, $\diamondsuit_\kappa(E)$ holds in L for any stationary subset E of any uncountable regular cardinal κ there. So if a cardinal κ (at least weakly Mahlo, so that REG is stationary) exhibits too large a large cardinal property, or if the universe is too close to L, then one should expect $\diamondsuit_\kappa(\text{REG})$ to hold. This article is about a sequence of results that squeeze between these two requirements, by aiming to produce failures of $\diamondsuit_\kappa(\text{REG})$ as high in the large cardinal hierarchy as possible.

Specifically, our main theorem continues the progression of two earlier results. First, Woodin [CW] showed that $\diamondsuit_\kappa(\text{REG})$ can fail at any weakly compact cardinal; in fact, the existence of a weakly compact cardinal is equiconsistent with the existence of a weakly compact cardinal κ such that $\neg\diamondsuit_\kappa(\text{REG})$, plus GCH. After this, Kai Hauser [Hau92] produced failures of $\diamondsuit_\kappa(\text{REG})$ at indescribable cardinals, where again the existence of a Π^m_n indescribable cardinal ($m, n \geq 1$) is equiconsistent with the existence of a Π^m_n indescribable cardinal κ such that $\neg\diamondsuit_\kappa(\text{REG})$, plus GCH. All these results address a question asked in Kanamori's article [Kan88], namely, if it is possible to have a large cardinal κ such that $\diamondsuit_\kappa(\text{REG})$ fails. Here, we push the violations of $\diamondsuit_\kappa(\text{REG})$ higher with the following:

Main Theorem *The existence of a strongly unfoldable cardinal is equiconsistent with the existence of a strongly unfoldable cardinal κ such that $\neg\diamondsuit_\kappa(\text{REG})$, plus* GCH.

In all three cases, what is actually proved is that if a cardinal κ is respectively weakly compact, indescribable or strongly unfoldable and the GCH holds, then this is preserved in a forcing extension in which $\neg\diamondsuit_\kappa(\text{REG})$ holds. A firm upper bound on how high in the large cardinal hierarchy one can have failures of $\diamondsuit_\kappa(\text{REG})$ is provided by the following easy observation about measurable cardinals. (Finer results were obtained by Jensen and Kunen, as we mentioned above.)

Observation 1 *If κ is a measurable cardinal, then $\diamondsuit_\kappa(\text{REG})$ holds.*

Proof: This well known argument follows the usual proof of $\diamondsuit$ in L. If A_α is defined for all $\alpha < \gamma$, then for regular γ let A_γ be any subset of γ such that $\{\, \alpha \in \text{REG} \cap \gamma \mid A_\gamma \cap \alpha = A_\alpha \,\}$ is not stationary in γ, if there is

any such set, and use any subset of γ otherwise. At every stage of the construction, therefore, we have added to the sequence $\vec{A}$ a set which has not yet been sufficiently anticipated, if such a set existed. If the resulting sequence $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ does not witness $\diamondsuit_\kappa(\text{REG})$, then there is a set $A \subseteq \kappa$ which is not anticipated on a stationary set of regular cardinals. Let $j : V \to M$ be the ultrapower embedding generated by a normal measure on κ, and consider the sequence $j(\vec{A})$. Specifically, let $A^* = j(\vec{A})(\kappa)$ be the κ^{th} set in this sequence. By definition, this set should be a subset of κ which is not anticipated by $j(\vec{A}) \upharpoonright \kappa = \vec{A}$ on a stationary set of regular cardinals in M, if there are any such sets in M. But since M and V have the same subsets of κ, the two models agree on the regular cardinals up to κ, on whether a set of such cardinals is stationary and on whether such a set is anticipated by $\vec{A}$. And since we assumed that there are subsets of κ that are not anticipated by $\vec{A}$ on a stationary set in V, it follows that there are such sets in M. Thus, in particular, the set $A^* = j(\vec{A})(\kappa)$ is not anticipated by $\vec{A}$ on a stationary set of regular cardinals. So there is a club set $C \subseteq \kappa$ such that if $\gamma \in C \cap \text{REG}$, then $A_\gamma \neq A^* \cap \gamma$. But since $\kappa \in j(C)$, this implies that $j(\vec{A})(\kappa) \neq j(A^*) \cap \kappa$, which is absurd since $j(A^*) \cap \kappa = A^* = j(\vec{A})(\kappa)$. $\Box$

If κ is measurable, then the diamond sequence $\vec{A}$ constructed above will witness $\diamondsuit_\kappa(\text{REG})$ in M, whenever $j : V \to M$ has critical point κ, and so the set of cardinals $\gamma < \kappa$ that are weakly compact and satisfy $\diamondsuit_\gamma(\text{REG})$ has measure one with respect to every normal measure on κ. The construction of Observation 1 has little to do with REG, and it produces a sequence $\vec{A}$ whose restriction to any set E in a normal measure on κ witnesses $\diamondsuit_\kappa(E)$.

One can also show that every measurable cardinal κ has a kind of Laver function $\ell : \kappa \to V_\kappa$ for weakly compact embeddings, meaning that for any $A \in H(\kappa^+)$ and any transitive set M of size κ containing A and κ, there is a weak compactness embedding $j : M \to N$ such that $j(\ell)(\kappa) = A$. This is called the Laver diamond principle for weak compactness and denoted $\LaTeX{}$-style $\diagup\!\!\!\!\lrcorner^{\text{wc}}_\kappa$ in [Ham], where similar notions are considered for a variety of large cardinals. All of the Laver diamond principles easily imply $\diamondsuit_\kappa(\text{REG})$.

The question of the exact boundary in the large cardinal hierarchy where $\diamondsuit_\kappa(\text{REG})$ can fail remains open. The results of this article squeeze this boundary to somewhere between the strongly unfoldable cardinals and the subtle cardinals.

We use Villaveces' [Vil98] embedding characterization of the strongly unfoldable cardinals, through which they resemble miniature strong cardinals

in the same way that embeddings help weakly compact cardinals to resemble miniature measurable cardinals. In each case, an embedding $j : V \to W$ defined on all of V is replaced with embeddings $j : M \to N$ defined only on a transitive structure of size κ. In our proof, however, we also show that strongly unfoldable cardinals have an embedding characterization analogous to the supercompact cardinals, making "miniature strong" equivalent to "miniature supercompact," a fact we found surprising.

2 Strongly Unfoldable Cardinals

The unfoldable and strongly unfoldable cardinals were introduced by Villaveces in [Vil98] as a direct generalization of the weakly compact cardinals. The embedding characterizations will be the most convenient: a cardinal κ is *weakly compact* if and only if for any transitive structure M of size κ containing κ as an element there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ. Since one can always restrict an embedding from a larger domain to a smaller domain, this is equivalent to insisting merely that every subset $B \subseteq \kappa$ can be placed into a structure M having an appropriate embedding $j : M \to N$. And of course there are such structures M such that (i) M is transitive of size κ, (ii) M models some fixed large finite fragment of set theory ZFC* and (iii) $M^{<\kappa} \subseteq M$. We will refer to M having these properties as the *κ-models* of set theory. Thus, κ is weakly compact if and only if every κ-model of set theory M has an embedding $j : M \to N$ into a transitive N with critical point κ. For the remainder of this article, when M is transitive and we say that there is an embedding $j : M \to N$, we mean to imply that N is also transitive, and we will usually also mean that the critical point of j is whatever cardinal κ is under consideration. Throughout we shall use the adjective inaccessible to mean a strongly inaccessible cardinal.

A cardinal κ is *θ-unfoldable* when every κ-model of set theory M has an embedding $j : M \to N$ with critical point κ and $j(\kappa) \geq \theta$. This is equivalent to insisting that every set $B \subseteq \kappa$ can be placed into a κ-model M having such an embedding. We will refer to such embeddings as the θ-unfoldability embeddings. The cardinal κ is *unfoldable* if it is θ-unfoldable for every θ. Generalizing this further, κ is *θ-strongly unfoldable* if every κ-model of set theory M has a θ-unfoldability embedding $j : M \to N$ that also satisfies $V_\theta \subseteq N$. And again, this is equivalent to requiring that any set $B \subseteq \kappa$ can

be placed into such an M. Finally, of course, κ is *strongly unfoldable* if it is θ-strongly unfoldable for every θ. The strongly unfoldable cardinals therefore look a bit like miniature strong cardinals, with embeddings defined only on structures of size κ, rather than the entire universe.

If $V = L$, the requirement that $V_\theta \subseteq N$ amounts merely to $L_{\beth_\theta} \subseteq N$, which is true of any N containing enough ordinals, and so in L every unfoldable cardinal is strongly unfoldable. Since every unfoldable cardinal is unfoldable in L (see [Vil98]), the two unfoldability notions have the same consistency strength. Nevertheless, results in [Ham01] show that the strong unfoldability of any unfoldable cardinal can be destroyed by forcing that preserves its unfoldability, so as large cardinal notions the two concepts are distinct.

The next two theorems show that the consistency strength of the existence of unfoldable or strongly unfoldable cardinals lies strictly between the totally indescribable cardinals and the subtle cardinals. Thus, our Main Theorem finds violations of $\diamondsuit_\kappa(\text{REG})$ higher in the large cardinal hierarchy than [Hau92].)

By [Hau91], a cardinal κ is Π^m_n-*indescribable* if for any κ-model M there is an embedding $j : M \to N$ with critical point κ such that N is Σ^m_n-correct, that is, $(V_{\kappa+m})^N \prec_n V_{\kappa+m}$ and $N^{|V_{\kappa+m-2}|} \subseteq N$ (meaning $N^{<\kappa} \subseteq N$ when $m = 1$). The usual reflection definition shows that it is equivalent to omit the latter closure requirement on N. One may assume that the embedding j comes from the ultrapower by an M-extender of length $|V_{\kappa+m-1}|$.

Theorem 2 *Every strongly unfoldable cardinal is totally indescribable and a limit of totally indescribable cardinals. Indeed, if κ is merely $(\kappa+\omega)$-strongly unfoldable, then the collection of totally indescribable cardinals below κ is stationary.*

Proof: Every strongly unfoldable cardinal is totally indescribable because the $(\kappa + m)$-strong unfoldability embeddings witness Π^m_n indescribability for any n. Suppose that κ is $(\kappa + \omega)$-strongly unfoldable and fix any club set $C \subseteq \kappa$. Let M be any κ-model of set theory with $C \in M$. Since M and V agree up to κ, they agree on the totally indescribable cardinals below κ. Fix any θ-strong unfoldability embedding $j : M \to N$ for any $\theta \geq \kappa + \omega$. Thus, $V_{\kappa+\omega} \subseteq N$. It follows that N has all κ-models and the extender embeddings on them to witness the Π^m_n indescribability of κ, and consequently κ is totally indescribable in N. Since κ is a limit point of $j(C)$, it follows that $\kappa \in$

$j(C)$ and so $j(C)$ meets the set of totally indescribable cardinals in N. By elementarity, C contains some totally indescribable cardinals in M and hence in V. So the set of such cardinals is stationary. $\square$

A cardinal κ is *subtle* if for any closed unbounded set $C \subseteq \kappa$ and any sequence $\langle A_\alpha \mid \alpha \in C\rangle$ with $A_\alpha \subseteq \alpha$, there is $\alpha < \beta$ in C with $A_\alpha = A_\beta \cap \alpha$. It is not difficult to see that any such cardinal is strongly inaccessible. We are grateful to Ralf Schindler for pointing out the following.

Theorem 3 *If κ is subtle, then the set of cardinals $\gamma < \kappa$ that are $<\kappa$-strongly unfoldable is stationary. Hence, V_κ has a stationary proper class of strongly unfoldable cardinals.*

Proof: Suppose not, so there is a closed unbounded set $C \subseteq \kappa$ containing no $<\kappa$-unfoldable cardinals. For each γ in C, there is some least $\lambda < \kappa$ such that γ is not λ-strongly unfoldable, and by thinning C we may assume that λ is less than the next element of C, and also that γ is a Beth fixed point. Since γ is not λ-strongly unfoldable, there is a transitive structure M_γ of size γ, with $\gamma \in M_\gamma$ and $V_\gamma \subseteq M_\gamma$, having no embedding $j : M_\gamma \to N$ with $\mathrm{cp}(j) = \gamma$, $j(\gamma) > \lambda$ and $V_\lambda \subseteq N$. Since M_γ has size γ, there is a relation E_γ on γ such that $\langle \gamma, E_\gamma\rangle \cong \langle M_\gamma, \in\rangle$. The isomorphism π_γ witnessing this is exactly the Mostowski collapse of $\langle \gamma, E_\gamma\rangle$. We may assume that $\pi_\gamma(0) = \gamma$. Let A_γ be a subset of γ coding E_γ, the elementary diagram of $\langle \gamma, E_\gamma\rangle$ and the map $\pi_\gamma^{-1} \restriction \gamma$.

Since κ is subtle, there is $\gamma < \delta$ in C with $A_\gamma = A_\delta \cap \gamma$. Define $j : M_\gamma \to M_\delta$ by $j = \pi_\delta \circ \pi_\gamma^{-1}$. Observe that $j(\gamma) = \pi_\delta(\pi_\gamma^{-1}(\gamma)) = \pi_\delta(0) = \delta$. Also, if $\alpha < \gamma$, then because A_γ and A_δ agree up to γ, it follows that $\pi_\gamma^{-1} \restriction \gamma = \pi_\delta^{-1} \restriction \gamma$, and so $j(\alpha) = \alpha$. So the critical point of j is γ. The map is elementary, since if $M_\gamma \models \varphi[x]$ where $x = \pi_\gamma(\alpha)$, then $\varphi(\alpha)$ is in the elementary diagram of $\langle \gamma, E_\gamma\rangle$, and so it is also in the elementary diagram of $\langle \delta, E_\delta\rangle$, which means $M_\delta \models \varphi[j(x)]$. Since $V_\delta \subseteq M_\delta$, we have found a δ-strong unfoldability embedding for M_γ, contradicting our assumptions. $\square$

Strongly unfoldable cardinals, like strong cardinals, have canonical extender embeddings, which have allowed for the borrowing of many techniques from the strong cardinal context. Lemma 5, however, shows that they also exhibit a supercompactness-like nature, allowing us to borrow techniques from the supercompact cardinal context as well. The fact that these miniature

versions of strong and supercompact cardinals are equivalent (see Corollary 7) is both interesting and surprising, so we include both characterizations.

Lemma 4 *If κ is strongly θ-unfoldable, then for any κ-model of set theory M there is a strong θ-unfoldability embedding $j : M \to N$ such that every object in N has the form $j(f)(\alpha)$ for some $\alpha < \beth_\theta$ and $f \in M \cap M^\kappa$.*

Proof: Suppose $j : M \to N$ is any strong θ-unfoldability embedding. Let $X = \{\, j(f)(\alpha) \mid f \in M \cap M^\kappa \;\;\&\;\; \alpha < \beth_\theta \,\}$. By verifying the Tarski-Vaught criterion, one can easily see that $X \prec N$, and in fact X is the (smallest) Skolem hull of $\mathrm{ran}(j)$ with the elements of $\beth_\theta$. If $j_0 = \pi \circ j$, where $\pi : X \cong N_0$ is the Mostowski collapse of X, then $j_0 : M \to N_0$ is an elementary embedding with critical point κ, and $j_0(\kappa) = \pi(j(\kappa)) \geq \sup \pi\,"\,\theta \geq \theta$. Let $s = \langle a_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of V_κ in M with the property that the objects of any V_β are enumerated before $\beth_\beta$. It follows that every object in $V_\theta = (V_\theta)^N$ is $j(s)(\alpha)$ for some $\alpha < \beth_\theta$, and so $V_\theta \subseteq X$. The elements of V_θ are therefore fixed by the collapse π, and so $V_\theta \subseteq N_0$. Finally, every object in N_0 has the form $\pi(j(f)(\alpha)) = j_0(f)(\pi(\alpha)) = j_0(f)(\alpha)$. So $j_0 : M \to N_0$ has the desired canonical form. $\square$

These embeddings are obtained by the ultrapowers by M-extenders, in analogy with strong cardinals. One could just as easily insist alternatively that every element of N had the form $j(f)(a)$ for some $f \in M \cap M^\kappa$ and $a \in V_\theta$. Next we turn to the supercompactness nature of strongly unfoldable cardinals.

Lemma 5 *If κ is $(\theta+1)$-strongly unfoldable, then for every κ-model of set theory M there is a $\beth_\theta$-closed $(\theta+1)$-strong unfoldability embedding $j : M \to N$. That is, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and*

1. *$V_{\theta+1} \subseteq N$,*
2. *$N^{\beth_\theta} \subseteq N$ and*
3. *$|N| = \beth_{\theta+1}$.*

Proof: Suppose that κ is $(\theta+1)$-strongly unfoldable and M is a κ-model of set theory. Since M has size κ, there is a relation E on κ such that there is an isomorphism $\pi : \langle \kappa, E \rangle \cong \langle M, \in \rangle$. This isomorphism π is necessarily

the Mostowski collapse of the structure $\langle \kappa, E\rangle$, and so it and the structure M are uniquely determined by E. By collapsing the Skolem hull of E in some very large V_λ, we may construct a κ-model $\overline{M}$ such that $E \in \overline{M}$. It follows that $M \in \overline{M}$ as well. By the $(\theta+1)$-strong unfoldability of κ, there is an embedding $j : \overline{M} \to \overline{N}$ with critical point κ such that $j(\kappa) > \theta$ and $V_{\theta+1} \subseteq \overline{N}$. Since $E = j(E) \cap \kappa \times \kappa$, it follows that $E \in \overline{N}$ and hence also $M \in \overline{N}$. Let $j_0 = j \restriction M$ and $N = j(M)$, so that $j_0 : M \to N$. The critical point of j_0 is still κ and $j_0(\kappa) = j(\kappa) > \theta$, so this is a $(\theta+1)$-unfoldability embedding. Since $M^{<\kappa} \subseteq M$ and $\overline{M}^{<\kappa} \subseteq \overline{M}$, it follows that M and $\overline{M}$ agree up to V_κ. Thus, $j(M)$ and $\overline{N}$ agree up to $(V_{j(\kappa)})^{\overline{N}}$. Since $\theta + 1 < j(\kappa)$, this includes the true $V_{\theta+1}$ and so $j_0 : M \to N$ is a $(\theta+1)$-strong unfoldability embedding.

Now we use Hauser's trick from his treatment of indescribable embeddings in [Hau91]. By elementarity, if $x \in M$ is coded by some $\alpha < \kappa$ with respect to E, then $j(x)$ is coded by $j(\alpha)$ with respect to $j(E)$. That is, if $x = \pi(\alpha)$ for some $\alpha < \kappa$, then $j(x) = j(\pi)(j(\alpha))$. But since $j_0(x) = j(x)$ and $j(\alpha) = \alpha$, this means that $j_0(x) = y$ if and only if x is coded by the same ordinal with respect to E that y is coded by with respect to $j(E)$. Consequently, the embedding j_0 is definable in $\overline{N}$ from E and $j(E)$ and their respective Mostowski collapses. Thus, the entire embedding $j_0 : M \to N$ is an element of $\overline{N}$.

Furthermore, since $\overline{M}$ knows that $M^{<\kappa} \subseteq M$, it follows that $\overline{N}$ satisfies $N^{<j(\kappa)} \subseteq N$. In particular, $N^{\beth_{\theta+1}} \subseteq N$ in $\overline{N}$. By a classical downward Löwenheim-Skolem argument in $\overline{N}$, we may build a set $X \prec N$ such that $\mathrm{ran}(j_0) \subseteq X$, $X^{\beth_\theta} \subseteq X$ and $|X| = \beth_{\theta+1}$ in $\overline{N}$. We observe that any function from $\beth_\theta$ to $V_{\theta+1}$ can be canonically coded by an element of $V_{\theta+1}$, and so because $\overline{N}$ has the true $V_{\theta+1}$, if $\overline{N}$ thinks that a set of size $\beth_{\theta+1}$, such as X, is closed under $\beth_\theta$ sequences, then $\overline{N}$ is correct about this. So actually we know that $X^{\beth_\theta} \subseteq X$ in V. Let $\pi^* : X \cong N^*$ be the Mostowski collapse of X and let $j^* = \pi^* \circ j_0$, so that $j^* : M \to N^*$. Since $X^{\beth_\theta} \subseteq X$, it follows that V_θ and all its subsets are in X. This implies $V_{\theta+1} \subseteq X$. These objects will therefore be fixed by π^*, and so $V_{\theta+1} \subseteq N^*$ as well. Since X has size $\beth_{\theta+1}$ and is isomorphic to N^*, it follows that N^* also has size $\beth_{\theta+1}$. Since $X^{\beth_\theta} \subseteq X$ and $X \cong N^*$, it follows that $N^{\beth_\theta} \subseteq N$. And finally, since $\pi^*(\theta) = \theta$, it follows that $j^*(\kappa) = \pi^*(j_0(\kappa)) \geq \pi^*(\theta+1) = \theta + 1$. Thus, the embedding $j^* : M \to N^*$ is a $\beth_\theta$-closed $(\theta+1)$-strong unfoldability embedding whose target is of size $\beth_{\theta+1}$, as we desired. $\square$

In particular, if the GCH holds at $\lambda = \beth_\theta$, then the lemma produces λ-strong unfoldability embeddings $j : M \to N$ such that $N^\lambda \subseteq N$ and $|N| = \lambda^+$. The same idea applies to the case when θ is a limit ordinal, but here we get only the $\mathrm{cof}(\theta)$ closure of N:

Lemma 6 *If κ is θ-strongly unfoldable for a limit ordinal θ, then for any κ-model of set theory M there is a θ-strong unfoldability embedding $j : M \to N$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and*

1. *$V_\theta \subseteq N$,*
2. *$N^{<\mathrm{cof}(\theta)} \subseteq N$ and*
3. *$|N| = \beth_\theta$.*

Proof: We proceed here just as above. Given M, we find E and $\overline{M}$ as above, and fix $j : \overline{M} \to \overline{N}$ a θ-strong unfoldability embedding. Once again, $j_0 = j \restriction M$ is in $\overline{N}$, and $\overline{N}$ sees that $N = j(M)$ is closed under $< j(\kappa)$ sequences. By a Löwenheim-Skolem argument, we find $X \prec N$ in $\overline{N}$ such that $V_\theta \subseteq X$, $X^{<\mathrm{cof}(\theta)} \subseteq X$ in $\overline{N}$ and $|X| = \beth_\theta$. Since $\overline{N}$ is correct about all the $< \mathrm{cof}(\theta)$ sequences in any set of size $\beth_\theta$, it follows that $X^{<\mathrm{cof}(\theta)} \subseteq X$ in V as well. And so if N^* is the Mostowski collapse of X, it follows again that the corresponding embedding $j^* : M \to N^*$ is as we desired. $\square$

Corollary 7 *The following are equivalent:*

1. *The cardinal κ is strongly unfoldable. That is, for every κ-model M and ordinal θ there is an embedding $j : M \to N$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and $V_\theta \subseteq N$.*
2. *For every κ-model M and ordinal θ there is an embedding $j : M \to N$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and $N^\theta \subseteq N$.*

So the miniature versions of strong and supercompact cardinals are equivalent. Most arguments in the literature involving strongly unfoldable cardinals have until now used the strongness-like extender embeddings, but the supercompactness-like characterization allows for simplifications, which we will make use of in our main argument.

3 Warming-Up with Just One Sequence

In this section, we present the one-step forcing for destroying a single potential $\diamondsuit_\kappa(\text{REG})$ sequence while preserving the strong unfoldability of κ.

Definition 8 A *potential* $\diamondsuit_\kappa$ *sequence* is a sequence $\vec{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$. For any such sequence, the *diamond killing poset* $\mathbb{Q}_{\vec{A}}$ has conditions (s, c) such that c is a closed bounded subset of κ, $s \subseteq \sup(c)$ and $A_\alpha \neq s \cap \alpha$ for every $\alpha \in c \cap \text{INACC}$. These conditions are ordered by end-extension above $\sup(c)$, that is: $(s', c') \leq (s, c)$ if and only $s' \cap \sup(c) = s$ and $c' \cap (\sup(c) + 1) = c$.

Potential $\diamondsuit_\kappa$ sequences are sometimes also referred to as 'ladder systems', but the present terminology seems more evocative in our case. The diamond killing poset clearly adds a set $S \subseteq \kappa$ and a club set $C \subseteq \kappa$ such that $A_\alpha \neq S \cap \alpha$ for any $\alpha \in C \cap \text{INACC}$, meaning that it has killed off $\vec{A}$ as a witness for $\diamondsuit_\kappa(\text{REG})$.

Lemma 9 *Suppose κ is inaccessible and $\vec{A}$ is a potential $\diamondsuit_\kappa$ sequence. Then:*

1. *The diamond killing poset $\mathbb{Q}_{\vec{A}}$ preserves all cardinals and cofinalities.*
2. *The diamond killing poset $\mathbb{Q}_{\vec{A}}$ adds no bounded sets to κ.*
3. *For every $\gamma < \kappa$, the diamond killing poset $\mathbb{Q}_{\vec{A}}$ has an open dense subset that is $\leq\gamma$-directed closed.*
4. *The diamond killing poset $\mathbb{Q}_{\vec{A}}$ is κ-centered and thus satisfies the κ^+ chain condition.*
5. *In the diamond killing extension $V^{\mathbb{Q}_{\vec{A}}}$, or any further forcing extension, the sequence $\vec{A}$ does not witness $\diamondsuit_\kappa(\text{REG})$.*

Proof: Statements 1 and 2 follow easily from 3 and 4. Statement 4 is immediate, because $\mathbb{Q}_{\vec{A}}$ has size κ. Statement 5 is also immediate from our observation that $\mathbb{Q}_{\vec{A}}$ adds a set S and a club set $C \subseteq \kappa$ such that $A_\alpha \neq S \cap \alpha$ for any $\alpha \in C \cap \text{INACC}$. So it remains only to prove statement 3. For this, fix any $\gamma < \kappa$ and let D be the set of conditions (s, c) such that $c \setminus \gamma$ is nonempty. This is certainly an open dense subset of $\mathbb{Q}_{\vec{A}}$, since we can strengthen any

condition to one mentioning ordinals above γ, and further strengthening only adds more elements to c. To see that D is $\leq\gamma$-directed closed, suppose that $\{ (s_\alpha, c_\alpha) \mid \alpha < \gamma \}$ is a directed family in D. Let $s = \bigcup_\alpha s_\alpha$ and $c = \bigcup_\alpha c_\alpha$ be the limiting values of the coherent sets, and let $\bar{c} = c \cup \{\sup(c)\}$ be the closure of c. Certainly $s \cap \delta = s_\alpha \cap \delta$ for any $\delta \in c_\alpha \cap \text{INACC}$ for large enough s_α, so $(s, \bar{c})$ satisfies the required nonreflection property $s \cap \delta \neq A_\delta$ for any $\delta \in c \cap \text{INACC}$. The top point $\sup(c)$ itself cannot be inaccessible, because it is above γ but has cofinality at most γ. So $(s, \bar{c})$ is a condition in D, and it is clearly below every (s_α, c_α). Statement 3 therefore holds. $\square$

The condition $(s, \bar{c})$ constructed as above will be referred to as the limit of $\{ (s_\alpha, c_\alpha) \mid \alpha < \gamma \}$, and denoting $q_\alpha = (s_\alpha, c_\alpha)$ and $q = (s, \bar{c})$, we may also say that $q = \inf_\alpha q_\alpha$.

While the diamond killing poset $\mathbb{Q}_{\vec{A}}$ kills off the sequence $\vec{A}$ as a witness for $\diamondsuit_\kappa(\text{REG})$, there are of course many other sequences. In order to kill $\diamondsuit_\kappa(\text{REG})$ fully, one expects to iterate this forcing, anticipating and killing all the potential diamond sequences that might arise. And with a wrinkle, this is how the proof of our main theorem will proceed.

Actually, if all one wants to do is kill the one sequence $\vec{A}$, there are much easier ways to do it. For example, by merely adding a single Cohen real one preserves the strong unfoldability of κ (and all other large cardinals) while destroying all diamond sequences in the ground model, because they do not anticipate the new real. But clearly we cannot hope to iterate this forcing to achieve our purposes. Rather, the diamond killing poset $\mathbb{Q}_{\vec{A}}$ is the natural choice for iterating when one wants not to add bounded subsets to κ.

Before we consider iterations of the diamond killing poset for different potential diamond sequences, we will analyze mere products of $\mathbb{Q}_{\vec{A}}$ with itself (where we kill the same sequence $\vec{A}$ on every coordinate; this is the wrinkle we mentioned above). Let $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ be the $<\kappa$-support product of κ^+ many copies of $\mathbb{Q}_{\vec{A}}$. It may seem odd to consider forcing that kills the same sequence $\vec{A}$ so many times, since one might expect that killing it once would be enough. But in order to preserve the strong unfoldability of κ in the extension, this many-times-over death of $\vec{A}$ as a witness for $\diamondsuit_\kappa(\text{REG})$ is what the argument seems to require. Our argument, specifically, will make critical use of the fact that we have κ^+ many different generic filters for $\mathbb{Q}_{\vec{A}}$ available to us. Hauser's argument [Hau92] for the Π^m_n indescribable cardinals exhibits the same repetitive killing feature, where he makes similar crucial use of the fact that every potential diamond sequence is killed unboundedly often during his

iteration. One difference in presentation is that while Hauser blends all his repetitions together in one long iteration, we have isolated the copies of $\mathbb{Q}_{\vec{A}}$ all in one product. This will allow us to illustrate the general technique as a warm-up in Theorem 11, in the simplified case of just one potential diamond sequence $\vec{A}$.

Lemma 10 *Suppose κ is inaccessible and $\vec{A}$ is a potential diamond sequence. Let $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ be the $<\kappa$-support product of the diamond killing poset $\mathbb{Q}_{\vec{A}}$. Then:*

1. *The forcing $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ preserves all cardinals and cofinalities.*
2. *The forcing $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ adds no bounded sets to κ.*
3. *For every $\gamma < \kappa$, the forcing $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ has a dense subset that is $\leq\gamma$-directed closed.*
4. *The forcing $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ is κ-centered and thus satisfies the κ^+ chain condition.*
5. *In the extension $V^{\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}}$, or any further forcing extension, the sequence $\vec{A}$ does not witness $\diamondsuit_\kappa(\text{REG})$.*

Proof: Again, statements 1 and 2 follow from 3 and 4. Statement 5 is immediate from Lemma 9, since $V^{\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}}$ is an extension of $V^{\mathbb{Q}_{\vec{A}}}$. For statement 3, we consider the set D consisting of conditions that are in the $\leq\gamma$-closed dense subset of $\mathbb{Q}_{\vec{A}}$ on each coordinate in their support. That is, D consists of conditions in the product such that every coordinate is either trivial or else mentions ordinals at least as large as γ. This is easily seen to be dense and $\leq\gamma$-directed closed by the same argument as in Lemma 9, so 3 holds. Statement 4 follows from the fact that $\kappa^{<\kappa} = \kappa$ and the poset is the $<\kappa$-support product of κ-centered forcing. $\Box$

We warn the reader that in Lemma 10, the $\leq\gamma$-directed closed dense set we produce in statement 3 is not open, as in Lemma 9, because one can always extend a condition by adding an element to the support and placing a condition there that does not jump up to γ.

For convenience, our main construction employs the lottery preparation of [Ham00]. While one could avoid this by tailoring the iteration below κ to ensure the desired reflectivity, the point of the lottery preparation is that

such efforts are unnecessary: one simply works below a condition opting for the desired forcing at stage κ, and the generic filter is thereby forced to have the desired reflectivity.

Let us quickly review the lottery preparation here. By forcing if necessary, we may assume that there is a function $f{:}\kappa \to \kappa$ with the strong unfoldability *Menas* property: for every κ-model M and every ordinal θ there is a θ-strong unfoldability embedding $j : M \to N$ such that $j(f)(\kappa) > \theta$. Arguments in [Ham01] show that such a function can be added by Woodin's fast function forcing, while preserving the strong unfoldability of κ. And it is easy to see that the collapse of X in Lemmas 5 and 6 will preserve the fact that $j(f)(\kappa) > \theta$, so we may take $j : M \to N$ to have the supercompactness-like form of those lemmas. The *lottery preparation* relative to f is the Easton support κ iteration $\mathbb{P}$ which at every stage $\gamma < \kappa$, provided that $\gamma \in \mathrm{dom}(f)$ and $f\,``\,\gamma \subseteq \gamma$, forces with the lottery sum of all $\mathbb{Q} \in H(f(\gamma))$ having for every $\beta < \gamma$ a dense subset that is $\leq\beta$-strategically closed. (The *lottery sum* $\oplus\mathcal{A}$ of a collection of posets $\mathcal{A}$, also commonly called side-by-side forcing, is the poset $\{\,\langle\mathbb{Q}, p\rangle \mid p \in \mathbb{Q} \in \mathcal{A}\,\} \cup \{\mathbb{1}\}$, ordered with $\mathbb{1}$ above everything and $\langle\mathbb{Q}, p\rangle \leq \langle\mathbb{Q}', p'\rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} p'$. The generic filter in effect selects a 'winning' poset from $\mathcal{A}$ and then forces with it.) By further restricting the lotteries to include only forcing notions that preserve the GCH, one obtains the GCH-*preserving* lottery preparation, and this preserves the GCH. The thrust of [Ham00] is that if j is an embedding with critical point κ, then the lottery sum at stage κ in $j(\mathbb{P})$ includes all the desired posets, and so by working below a condition opting for the correct poset in that lottery, one avoids the need for a Laver function. Arguments in [Ham01] show that the lottery preparation of a strongly unfoldable cardinal κ preserves the strong unfoldability of κ, and makes the strong unfoldability of κ indestructible, for example, by the forcing $\mathrm{add}(\kappa, 1)$, among others. And these arguments are easily adapted to the GCH-preserving lottery preparation.

We turn now to the Warm-Up Theorem, which shows how to kill off a single potential diamond sequence $\vec{A}$, while preserving the strong unfoldability of κ and adding no new bounded sets to κ. Later, in the Main Theorem, we will simply iterate this forcing so as to anticipate all possible sequences $\vec{A}$, thereby forcing the negation of $\Diamond_\kappa(\mathrm{REG})$ while preserving the strong unfoldability of κ.

The Warm-Up Theorem 11 *Suppose that κ is strongly unfoldable in V, the* GCH *holds and $V[G]$ is the* GCH*-preserving lottery preparation of κ relative to $f{:}\kappa \to \kappa$. Then for any potential diamond sequence $\vec{A}$ in $V[G]$, the cardinal κ remains strongly unfoldable after further forcing with $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$.*

Proof: Suppose that $g \subseteq \Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ is $V[G]$-generic for the $<\kappa$-support κ^+ product of the diamond killing poset $\mathbb{Q}_{\vec{A}}$ corresponding to $\vec{A} = \langle A_\gamma \mid \gamma < \kappa \rangle$. This forcing preserves the GCH because it adds no bounded subsets to κ and has size κ^+. The generic filter g is determined by the generic objects (S_ξ, C_ξ) added by the copy of $\mathbb{Q}_{\vec{A}}$ at each coordinate $\xi < \kappa^+$. Thus, $C_\xi \subseteq \kappa$ is a club set, $S_\xi \subseteq \kappa$ and for every $\gamma \in C_\xi \cap \text{INACC}$ we have $S_\xi \cap \gamma \neq A_\gamma$.

We claim that κ is strongly unfoldable in $V[G][g]$. If not, then there is a condition $p \in g$, an ordinal $\theta > \kappa$ and a name $\dot{B}$ for a subset of κ such that in $V[G]$, the condition p forces that $\dot{B}$ cannot be placed into a κ-model having a θ-strong unfoldability embedding. By increasing θ if necessary, we may assume that $\theta = \beth_\theta$. Because the forcing is κ^+-c.c., we may assume that the name $\dot{B}$ has hereditary size at most κ and depends on at most κ many coordinates of the product forcing.

Let M be a κ-model of set theory in V, large enough to contain the things (of size κ) on which we are focused: κ, f, $\mathbb{P}$ and names for $\vec{A}$, $\dot{B}$ and p, so that these latter items are elements of $M[G]$. Because κ is strongly unfoldable in V and the function f has the Menas property, by Lemma 5 there is a $(\theta+1)$-strong unfoldability embedding $j : M \to N$ in V with critical point κ such that $j(f)(\kappa) > \theta$, $N^\theta \subseteq N$, $V_{\theta+1} \subseteq N$ and $|N| = \theta^+$. Since $V_\theta \subseteq N$, the model N has the same κ^+ as V, and so the full forcing $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ appears as one of the choices in the stage κ lottery of $j(\mathbb{P})$. Below the condition opting for this poset, the forcing $j(\mathbb{P})$ factors as $\mathbb{P} * (\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}) * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing from stages $\kappa+1$ up to $j(\kappa)$. Note that $G * g$ is N-generic for the first $\kappa+1$ many stages of this forcing. Since $j(f)(\kappa) > \theta$, the next nontrivial stage of forcing in $\mathbb{P}_{\text{tail}}$ is beyond θ, and so $\mathbb{P}_{\text{tail}}$ has a $\leq\theta$-strategically closed dense set in $N[G][g]$. Furthermore, there are only θ^+ many elements in N altogether, so in $V[G][g]$ we may list the dense subsets of $\mathbb{P}_{\text{tail}}$ in $N[G][g]$ in a θ^+-sequence. Since $N^\theta \subseteq N$ in V and the forcing $G * g$ is κ^+-c.c., it follows that $N[G][g]^\theta \subseteq N[G][g]$ in $V[G][g]$. We may therefore construct by diagonalization in $V[G][g]$ a descending sequence of conditions in $\mathbb{P}_{\text{tail}}$ that meet every dense set in $N[G][g]$. These generate therefore an $N[G][g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$, and we may lift the embedding to $j : M[G] \to N[j(G)]$, where $j(G) = G * g * G_{\text{tail}}$. Since $\vec{A}$ has a name in M, it follows that $\vec{A} \in M[G]$

and so $j(\vec{A})$ is a $j(\kappa)$-sequence in $N[j(G)]$.

Consider now the forcing $(\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}})^{M[G]}$, which is the same as $\Pi_\delta\mathbb{Q}_{\vec{A}}$ in $V[G]$, where $\delta = (\kappa^+)^M$. It follows that $g \upharpoonright \delta$ is $M[G]$-generic for this forcing. The lucky case occurs if it happens that $S_\alpha \neq j(\vec{A})(\kappa)$ for all $\alpha < \delta$. In this case, we use $j\,"\,\delta \in N$ to build the (master) condition $q \in j(\Pi_\delta\mathbb{Q}_{\vec{A}})$ having support $j\,"\,\delta$, with the pair $(S_\alpha, C_\alpha \cup \{\kappa\})$ at coordinate $j(\alpha)$. First of all, this object is an element of $N[j(G)]$ because $g \in N[j(G)]$ and $j\,"\,\delta \in N$. It is a condition in $j(\Pi_\delta\mathbb{Q}_{\vec{A}})$ because it has the correct support, it satisfies all the required non-reflection conditions below κ, and it satisfies the non-reflection condition at stage κ by the lucky property that $j(S_\alpha) \cap \kappa = S_\alpha \neq j(\vec{A})(\kappa)$. This condition is called a master condition because it is stronger than every element of $j\,"\,(g \upharpoonright \delta)$. Below this (or any) condition, the forcing $j(\Pi_\delta\mathbb{Q}_{\vec{A}})$ has a $\leq\theta$-strategically closed dense set, so using the fact that $N[j(G)]^\theta \subseteq N[j(G)]$ in $V[G][g]$ we may once again diagonalize to build a $N[j(G)]$-generic filter containing q. The embedding therefore lifts in $V[G][g]$ to $j : M[G][g \upharpoonright \delta] \to N[j(G)][j(g \upharpoonright \delta)]$. And since $V_{\theta+1} \subseteq N$ and $G * g \in N[j(G)]$, it follows that $(V[G][g])_{\theta+1} \subseteq N[j(G)]$, making the lift a strong $(\theta+1)$-unfoldability embedding in $V[G][g]$. Finally, since $\dot{B} \in M[G]$, the value of $\dot{B}_g$ depends only on the first δ many coordinates of g, so $B = \dot{B}_g = \dot{B}_{g\upharpoonright\delta} \in M[G][g \upharpoonright \delta]$, which contradicts our assumption that there was no θ-strongly unfoldable embedding for B.

In general, however, we may not be in the lucky case (see Remark 12), and our task is a bit harder. Suppose the unlucky case occurs, and $S_{\alpha_0} = j(\vec{A})(\kappa)$ for some $\alpha_0 < \delta$. Since the S_α are distinct, this occurs for at most one $\alpha_0 < \kappa^+$. Consider the original condition p, forcing that $\dot{B}$ named a counterexample, and it's value $p(\alpha_0)$ on coordinate α_0. It is dense that there is some other coordinate $\alpha_1 \geq \delta$ such that the generic object on coordinate α_1 extends $p(\alpha_0)$ (after all, any given condition in $\mathbb{Q}_{\vec{A}}$ will occur in the product generic on unboundedly many coordinates). Let π^* be the automorphism of $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}}$ determined by swapping coordinates α_0 and α_1. If $g^* = \pi^*\,"\,g$, then our choice of α_1 ensures that $p \in g^*$. Furthermore, because the automorphism is in the ground model, it follows that $V[G][g] = V[G][g^*]$. And finally, since $S_{\alpha_1} \neq S_{\alpha_0} = j(\vec{A})(\kappa)$, the generic g^* is in the lucky case when it comes to lifting the embedding $j : M[G] \to N[j(G)]$ that we have already constructed so far. So as above we may construct a master condition below $j\,"\,g^* \upharpoonright \delta$ and an $M[j(G)]$-generic filter containing it, thereby lifting the embedding to $j : M[G][g^* \upharpoonright \delta] \to N[j(G)][j(g^* \upharpoonright \delta)]$. Since we arranged that $p \in g^*$

and $B^* = \dot{B}_{g^*} \in M[G][g^* \restriction \delta]$, this contradicts once again our assumption that p forced that $\dot{B}$ named a set that could not be placed into a θ-strong unfoldability embedding. $\square$

Remark 12 We emphasize that in the argument above, we are not claiming that every generic g has an isomorphic copy g^* placing $V[G][g^*]$ into the lucky case. This is actually false, because the sequence $\vec{A}$ could be defined so that A_γ picks out the first generic set $S_0 \subseteq \gamma$ added by the forcing G at stage γ, if the forcing G opted at stage γ for a γ^+ product of diamond killing posets. In this case, if one runs the construction from scratch in $V[G][g^*]$, we will have $j(G) = G * g^* * G_{\text{tail}}$ and consequently $j(\vec{A})(\kappa) = S_0^*$, the first set added by g^*, placing us in the unlucky case regardless of which g^* we use. The point of the argument, rather, is to use the original generic g to build the partial lift $j : M[G] \to N[j(G)]$ with $j(G) = G * g * G_{\text{tail}}$, thereby fixing the value of $j(\vec{A})(\kappa)$, and then to shift midstream to an isomorphic copy g^* that puts us into the lucky case for the rest of the argument. The remarks at the end of this article show that in general there can be no way for a pure lifting argument to succeed when one is trying to destroy $\diamondsuit_\kappa(\text{REG})$, because such arguments will preserve the existence of a strong unfoldability Laver function, which implies $\diamondsuit_\kappa(\text{REG})$.

4 The Main Theorem

We will now iterate the one-step forcing of the previous section in order to kill off all the potential $\diamondsuit_\kappa(\text{REG})$ sequences that may arise.

Main Theorem 13 *If κ is strongly unfoldable, then there is a forcing extension preserving this in which $\diamondsuit_\kappa(\text{REG})$ fails and the* GCH *holds.*

Proof: The main result of [Ham01] shows in part that by forcing if necessary, we may assume that the GCH holds and there is a function f with the Menas property for strong unfoldability. Suppose $G \subseteq \mathbb{P}$ is V-generic for the GCH-preserving lottery preparation of κ with respect to f. Let $\mathbb{Q}$ be the ${<}\kappa$-support forcing iteration which at each stage $\nu < \kappa^+$ forces with the κ^+ product $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}^\nu}$ for some potential diamond sequence $\vec{A}^\nu = \langle A^\nu_\gamma \mid \gamma < \kappa\rangle$, chosen by a book-keeping function in such a way that all potential diamond sequences added by $\mathbb{Q}$ are handled by some stage. Such a book-keeping

function exists because the iteration $\mathbb{Q}$, being the $<\kappa$-support iteration of κ-centered forcing at each stage, is κ-centered and hence κ^+-c.c.; so any potential diamond sequence will appear by some stage of the forcing. And since any such stage of the forcing adds at most $2^\kappa = \kappa^+$ such sequences, we can anticipate all the sequences that arise in one κ^+-iteration. Suppose $H \subseteq \mathbb{Q}$ is $V[G]$-generic for this forcing. The generic object H is determined, therefore, by the sequence of sequences $\langle\langle(S^\nu_\alpha, C^\nu_\alpha) \mid \alpha < \kappa^+\rangle \mid \nu < \kappa^+\rangle$, where $(S^\nu_\alpha, C^\nu_\alpha)$ are the sets added at the α^{th} coordinate of the product in the stage ν forcing. By the properties identified in Lemma 10 for each step of the iteration, it follows that the iteration altogether preserves all cardinals and cofinalities, adds no bounded subsets to κ, has for any $\gamma < \kappa$ a $\leq\gamma$-directed closed dense set and satisfies the κ^+-chain condition. Because the book-keeping function $\langle \vec{A}^\nu \mid \nu < \kappa^+\rangle$ eventually considers every potential diamond sequence in $V[G][H]$, killing $\vec{A}^\nu$ at stage ν, it follows that $V[G][H] \models \neg\Diamond_\kappa(\text{REG})$. Finally, the GCH-preserving lottery preparation preserves the GCH to $V[G]$, and the remaining forcing has size κ^+ and adds no bounded subsets to κ, so $V[G][H]$ satisfies the GCH.

It remains to show that κ remains strongly unfoldable in $V[G][H]$. Suppose towards contradiction that it does not. Then there is a condition $p \in H$, an ordinal $\theta > \kappa$ and a name $\dot{B}$ for a subset of κ such that in $V[G]$ the condition p forces that $\dot{B}$ is not captured by any θ-strong unfoldability embedding. We may assume as in the warm-up argument that $\theta = \beth_\theta$ and that $\dot{B}$ is a nice name. By the chain condition and the support in the iteration $\mathbb{Q}$, it follows that $\dot{B}$ mentions at most κ many stages of $\mathbb{Q}$ and within any such stage, mentions at most κ many coordinates (recall that each stage ν of the forcing is the $<\kappa$-support product $\Pi_{\kappa^+}\mathbb{Q}_{\vec{A}^\nu}$ of the diamond killing poset $\mathbb{Q}_{\vec{A}^\nu}$). The name $\dot{B}$ therefore has hereditary size at most κ. Let $\lambda \gg \theta$ be an enormous cardinal such that V_λ is a model of set theory, and construct $X \prec V_\lambda$ of size κ in V such that $X^{<\kappa} \subseteq X$ and X contains all of the objects on which we are focused: κ, $\mathbb{P}$, $\mathbb{Q}$, $\dot{B}$, p and the book-keeping function $\langle\vec{A}^\nu \mid \nu < \kappa^+\rangle$. Let M be the Mostowski collapse of X. Because $X^{<\kappa} \subseteq X$ and $X \prec V_\lambda$, it follows that M is a κ-model of set theory. Furthermore, since the objects of hereditary size κ are not affected by the collapse of X to M, we know that κ, $\mathbb{P}$, p and $\dot{B}$ are in M. Since G is V-generic, it is also M-generic and we may form the extension $M[G]$, the GCH-preserving lottery preparation of κ with respect to f. Since the book-keeping function is collapsed to the initial segment of itself of length $(\kappa^+)^M$, the iteration $\mathbb{Q}^{M[G]}$ is the $(\kappa^+)^M$ iteration

which at stage $\nu < (\kappa^+)^M$ forces with the product of $(\kappa^+)^M$ many copies of the diamond killing poset $\mathbb{Q}_{\vec{A}^\nu}$. Thus, $\mathbb{Q}^{M[G]}$ is completely embedded in $\mathbb{Q}$, and the restriction of H to the appropriate domains will be $V[G]$-generic for $\mathbb{Q}^{M[G]}$.

Let $j : M \to N$ be a $(\theta+1)$-strong unfoldability embedding in V witnessing the Menas property of f, so that $j(f)(\kappa) > \theta$, $N^\theta \subseteq N$ in V, $V_{\theta+1} \subseteq N$ and $|N| = \theta^+$. As in the warm up argument of Theorem 11, we may lift the embedding in $V[G][H]$ to $j : M[G] \to N[j(G)]$, where $j(G) = G * H * G_{\text{tail}}$. (This proceeded, as you recall, by opting for the full iteration $\mathbb{Q}$ in the stage κ lottery of $j(\mathbb{P})$, using the generic object H at this stage and building the remainder of the generic G_{tail} by diagonalization.) We will now lift the embedding through the forcing $\mathbb{Q}^{M[G]}$. As in the warm-up argument, while the restriction of H does provide an $M[G]$-generic filter for this forcing, it may not admit a master condition that allows us to lift the embedding. As before, we will instead build a master condition for a suitable isomorphic copy of H that does admit a master condition.

Lemma 13.1 *There is an automorphism $\pi : \mathbb{Q} \cong \mathbb{Q}$ and a condition $q \in j(\mathbb{Q}^{M[G]})$ such that if $H^* = \pi\,``\,H$, then $p \in H^*$ and $q$ is below every element of $j\,``\,H^* \restriction \mathbb{Q}^{M[G]}$.*

Proof: The automorphism π will be essentially an iteration of the kinds of automorphisms that arose in Theorem 11, which merely swapped two coordinates in a large product. Here, at each stage $\nu < (\kappa^+)^M$, the automorphism π will either do nothing to the stage ν forcing or else swap two coordinates in the product at stage ν. At stages beyond $(\kappa^+)^M$, the automorphism will do nothing extra (though of course any automorphism of the earlier stages naturally induces a map in the names forming the later stages, which we intend to be carried out without further remarks). The master condition q will be the limit of a descending sequence of conditions q_β for $\beta < (\kappa^+)^M$ that we construct simultaneously along with π. Each of those proto-master conditions q_β will be chosen from the fixed $\leq\kappa$-closed dense set $D \subseteq j(\mathbb{Q}^{M[G]})$ consisting as in Lemma 10 of those conditions which at any stage and on any coordinate in the product at that stage, are (with Boolean value 1) either trivial or else mention ordinals at least as large as κ. Thus, D is the iteration of the sets shown in Lemma 10 to be $\leq\kappa$-closed at each stage, and so it is $\leq\kappa$-closed in $N[j(G)]$, and the restriction of D to any stage of forcing is also $\leq\kappa$-closed.

Suppose now that the conditions q_β have been chosen for all $\beta < \nu$, where $\nu < (\kappa^+)^M$, and that the action of the automorphism π has been specified on $\mathbb{Q} \upharpoonright \nu$, the iteration of all the stages of forcing up to ν. We assume inductively that $q_\beta \in j(\mathbb{Q}^{M[G]}) \upharpoonright j(\beta)$, so that $q^* = \inf_{\beta<\nu} q_\beta \in j(\mathbb{Q}^{M[G]}) \upharpoonright \sup j\,"\,\nu$. The lucky case occurs, as in Theorem 11, when there is a condition q^{**} below q^* in $D \cap j(\mathbb{Q}^{M[G]}) \upharpoonright j(\nu)$ forcing that $j(\vec{A}^\nu)_\kappa$ is not equal to S^ν_α for any $\alpha < (\kappa^+)^M$. In this case, we let $q_\nu = q^{**} \frown r$ where r is the name of the condition at stage $j(\nu)$ having support $j\,"\,(\kappa^+)^M$ in the $j((\kappa^+)^M)$ product at that stage, and placing $(S^\nu_\alpha, \bar{C}^\nu_\alpha)$ at coordinate $j(\alpha)$, where $\bar{C}^\nu_\alpha = C^\nu_\alpha \cup \{\kappa\}$. That is, $q_\nu(j(\nu), j(\alpha)) = r(j(\alpha)) = (S^\nu_\alpha, \bar{C}^\nu_\alpha)$. Note that this is a condition, because the support is smaller than $j(\kappa)$, and the lucky condition guarantees exactly the nonreflection property we need at $\kappa \in \bar{C}^\nu_\alpha$. Also, since $\kappa \in \bar{C}^\nu_\alpha$, this condition remains in the dense set D. Finally, in this lucky case, the automorphism does nothing to the stage ν forcing.

In the unlucky case, there will be a condition $q^{**} \leq q^*$ in $D \cap j(\mathbb{Q}^{M[G]}) \upharpoonright j(\nu)$ and an ordinal $\alpha_0 < (\kappa^+)^M$ such that q^{**} forces $j(\vec{A}^\nu)_\kappa = S^\nu_{\alpha_0}$. As in Theorem 11, we may find an ordinal $\alpha_1 \geq (\kappa^+)^M$ and a still stronger condition q^{***} forcing that the value of $p(\nu, \alpha_0)$ is included in $(S^\nu_{\alpha_1}, C^\nu_{\alpha_1})$. In this case, we let $q_\nu = q^{***} \frown r$ where again $r(j(\nu), j(\alpha)) = (S^\nu_\alpha, \bar{C}^\nu_\alpha)$ for $\alpha \neq \alpha_0$, but $r(j(\nu), j(\alpha_0)) = (S^\nu_{\alpha_1}, \bar{C}^\nu_{\alpha_1})$. Let the automorphism π act on the stage ν forcing by swapping the coordinates α_0 and α_1 in the product at stage ν. This completes the recursive definition of the q_β and of π. Let q be the limit of the conditions q_β for $\beta < (\kappa^+)^M$. This limit exists because $(\kappa^+)^M < \kappa^+$ and D is $\leq\kappa$-closed.

Let H^* be the image of H under the automorphism π. Observe that by the choice of α_1 at stage ν, it follows that $H(\nu, \alpha_1)$ extends $p(\nu, \alpha_0)$, and so $H^*(\nu, \alpha_0)$ extends $p(\nu, \alpha_0)$. Since on the other coordinates, H^* and H agree, it follows that $p \in H^*$. Finally, since by construction the condition at $q(j(\nu), j(\alpha))$ for $\nu, \alpha < (\kappa^+)^M$ extends every condition in the generic at $H^*(\nu, \alpha)$, it follows that q is below every condition in $j\,"\,H^* \upharpoonright \mathbb{Q}^{M[G]}$, as desired. $\Box$

To complete the proof of the theorem, given the lemma, we proceed as in the warm-up of Theorem 11. Since q is a master condition for the forcing $H^* \upharpoonright \mathbb{Q}^{M[G]}$, we can use the diagonalization technique to lift the embedding fully to $j : M[G][H^* \upharpoonright \mathbb{Q}^{M[G]}] \to N[j(G)][j(H^* \upharpoonright \mathbb{Q}^{M[G]})]$. As before, this is a $(\theta + 1)$-strong unfoldability embedding in $V[G][H] = V[G][H^*]$. If $B = \dot{B}_{H^*} = \dot{B}_{H^* \upharpoonright \mathbb{Q}^{M[G]}}$, then as before it follows that $B \in M[G][H^* \upharpoonright \mathbb{Q}^{M[G]}]$, and

so this embedding has captured B. But the condition p forced that $\dot{B}$ named a set that could not be captured by any θ-strong unfoldability embedding. This contradicts the fact that $p \in H^*$. $\Box$

Corollary 14 *The existence of an unfoldable cardinal is equiconsistent with the existence of a strongly unfoldable cardinal κ satisfying $\neg\Diamond_\kappa(\mathrm{REG})$, plus the* GCH.

The proof of the main theorem can be refined to give the following more local result.

Theorem 15 *If κ is $(\theta+1)$-strongly unfoldable, then there is a forcing extension preserving this in which $\neg\Diamond_\kappa(\mathrm{REG})$ holds.*

Proof: The point is that in the proof above, we do not really need to assume that $\theta = \beth_\theta$. The argument works perfectly well just knowing that the GCH holds at $\beth_\theta$. And one can force this by simply adding a Cohen subset to $\beth_\theta^+$. This forcing will ensure that $\beth_{\theta+1} = \beth_\theta^+$, and since it is $\leq \beth_\theta$-closed, it will add no new κ-models. One can now run the argument above, lifting the embedding $j : M \to N$, but modified to place the Cohen subset of $\beth_\theta^+$ in the stage κ forcing of $j(\mathbb{P})$, as well as the κ^+ iteration $\mathbb{Q}$. Since N has size $\beth_\theta^+$ and is $\leq \beth_\theta$-closed, the diagonalization arguments are still able to produce the tail generic filters, so that j lifts to $j : M[G] \to N[j(G)]$ with $j(G) = G*(H*A)*G_{\mathrm{tail}}$, where $H \subseteq \mathbb{Q}$ is the generic for the diamond killing iteration, A is the Cohen subset of $\beth_\theta^+$ and G_{tail} is constructed by diagonalization. The master condition and automorphism argument go through unchanged to produce $j : M[G][H] \to N[j(G)][j(H)]$. This will be a $(\theta+1)$-strong unfoldability embedding because $V_{\theta+1} \subseteq N$ and we placed $A * H$ at stage κ in $j(\mathbb{P})$. Since this embedding captures the set B, whose name we placed into M as above, we conclude that κ is $(\theta+1)$-strongly unfoldable in $V[G][H][A]$. (Note that there is no need to lift the embedding through the forcing to add A, because every subset of κ is contained in a κ-model in $V[G][H]$.) $\Box$

5 Further Observations

We will now show how to apply our analysis to separate two intimately connected combinatorial principles: the existence of an ordinal anticipating

Laver function versus a full set-anticipating Laver function, in the large cardinal contexts from weakly compact to strong unfoldable. The fact that these are not equivalent answers an open question in [Ham].

Definition 16 ([Ham]) The *Laver Diamond principle* $\LaverDiamond^{\rm wc}_\kappa$ for weak compactness is the assertion that there is a function $\ell : \kappa \to V_\kappa$ such that for any $A \in H(\kappa^+)$ and any κ-model M containing ℓ and A there is an embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = A$. (An equivalent principle is investigated independently in [SV] and [Hel03]. The principle implies $\Diamond_\kappa(\textsc{reg})$ because any club set $C \subseteq \kappa$ can be placed along with A into such an M, and so from $j(\ell)(\kappa) = A$ and $\kappa \in j(C)$ it follows that ℓ anticipates A in $C \cap \textsc{reg}$.) The function ℓ is a mere *ordinal-anticipating* Laver function if for any κ-model M containing ℓ and any ordinal $\alpha < \kappa^+$ there is an embedding $j : M \to N$ such that $j(\ell)(\kappa) = \alpha$. The Laver Diamond principle $\LaverDiamond^{\rm sunf}_\kappa$ for strong unfoldability asserts that there is a function $\ell : \kappa \to V_\kappa$ such that for any ordinal θ, any $A \in V_\theta$ and any κ-model M containing ℓ, there is a θ-strong unfoldability embedding $j : M \to N$ such that $j(\ell)(\kappa) = A$. An *ordinal-anticipating* strong unfoldability Laver function is only required to anticipate every ordinal $\alpha < \theta$ with $j(\ell)(\kappa) = \alpha$. One can similarly define the Laver Diamond principle $\LaverDiamond^{\rm ind}_\kappa$ for indescribability and $\LaverDiamond^{\rm unf}_\kappa$ for unfoldability, as well as for many other large cardinals, along with the ordinal-anticipating versions of these principles.

Theorem 17 *If V has a Laver Diamond $\LaverDiamond^{sunf}_\kappa$ function, then the extension $V[G][H]$ of the Main Theorem nearly does. Specifically, there is a function $\ell^* : \kappa \to V[G]_\kappa$ in $V[G][H]$ with the following properties for any θ:*

1. *For any $B \in P(\kappa)^{V[G]}$ and any $A \in V_\theta[G][H]$, there is a κ-model M with $B, \ell^* \in M$ and a θ-strong unfoldability embedding $j : M \to N$ with $j(\ell^*)(\kappa) = A$.*
2. *For any $B \in P(\kappa)^{V[G][H]}$ and any $A \in V_\theta[G]$, there is a κ-model M with $B, \ell^* \in M$ and a θ-strong unfoldability embedding $j : M \to N$ with $j(\ell^*)(\kappa) = A$.*

Proof: Suppose that ℓ is a $\LaverDiamond^{\rm sunf}_\kappa$ Laver Diamond function in V. Such a function has the Menas property, in the sense that $j(\ell)(\kappa)$ can be made to have as large rank as desired, so we will assume that the lottery preparation

has nontrivial forcing only at stages γ for which $\ell\text{"}\gamma \subseteq V_\gamma$. Let $\ell^*(\alpha) = x_{G_{\alpha+1}}$ when $\ell(\alpha) = \langle x, y\rangle$ for some y and x is a $\mathbb{P}_{\alpha+1}$-name. For statement 1, suppose that $B \in P(\kappa)^{V[G]}$ and $A \in V_\theta[G][H]$. Let $\dot B$ be a name for B of hereditary size at most κ, and let M be any κ-model in V with $\dot B, \ell, \mathbb{P} \in M$. Let $\dot A$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for A, and fix any θ-strong unfoldability embedding $j : M \to N$ with $j(\ell)(\kappa) = \langle \dot A, \theta\rangle$. The θ in the second coordinate ensures that the next nontrivial stage of forcing is beyond θ, and so the construction of the Main Theorem shows how to lift the embedding to $j : M[G][H^* \cap \mathbb{Q}^{M[G]}] \to N[j(G)][j(H^* \cap \mathbb{Q}^{M[G]})]$, where H^* is an isomorphic image of $H \subseteq \mathbb{Q}$. Since $\dot B \in M$ it follows that $B \in M[G]$. And since the construction has $j(G)_{\kappa+1} = G * H$ and $j(\ell)(\kappa) = \langle \dot A, \theta\rangle$, it follows by the definition of ℓ^* that $j(\ell^*)(\kappa) = \dot A_{G*H} = A$. So the lifted embedding has all the features we desired for statement 1.

For statement 2, we are a bit more careful. Fix $B \in P(\kappa)^{V[G][H]}$ and $A \in V_\theta[G]$, with respective names $\dot A$ and $\dot B$ in V. Let $\check{\dot A}$ be the result of viewing $\dot A$ as a $\mathbb{P} * \dot{\mathbb{Q}}$-name, rather than merely a $\mathbb{P}$-name. If there is no such θ-strong unfoldability embedding as in Statement 2, then there will be a condition $p \in \mathbb{Q}$ forcing over $V[G]$ that $\dot B$ cannot be placed into a κ-model having a θ-strong unfoldability embedding for which $j(\ell^*)(\kappa) = \check A$. Now fix as above any κ-model M containing $\dot B, \ell, \mathbb{P}$ and a θ-strong unfoldability embedding $j : M \to N$ for which $j(\ell)(\kappa) = \langle \check{\dot A}, \theta\rangle$. The construction of the main theorem lifts this embedding to $j : M[G][H^* \cap \mathbb{Q}^{M[G]}] \to N[j(G)][j(H^* \cap \mathbb{Q}^{M[G]})]$, where H^* is an isomorphic image of $H \subseteq \mathbb{Q}$, such that $p \in H^*$. Since $j(G)_{\kappa+1} = G * H$, it follows once again that $j(\ell^*)(\kappa) = (\check{\dot A})_{G*H} = \dot A_G = A$. Let $B^* = \dot B_{G*H^*}$, which is determined of course by $G * (H^* \cap \mathbb{Q}^{M[G]})$. Thus, we have constructed in $V[G][H] = V[G][H^*]$ a κ-model $M[G][H^* \cap \mathbb{Q}^{M[G]}]$ containing B^* with an embedding for which $j(\ell^*)(\kappa) = A$. This contradicts the fact that $p \in H^*$, since p forced that there were no such model and embedding. $\square$

It is not possible to combine statements 1 and 2 in our model, and have that for every $B \subseteq \kappa$ in $V[G][H]$ and every $A \in V_\theta[G][H]$ there is a κ-model M containing B and ℓ^*, with a θ-strong unfoldability embedding $j : M \to N$ such that $j(\ell^*)(\kappa) = A$. The reason this is not possible is that such a statement would exactly assert the Laver Diamond $\mathbb{L}_\kappa^{\text{sunf}}$ principle in $V[G][H]$, and this implies $\Diamond_\kappa(\text{REG})$, which fails in $V[G][H]$. The point of Theorem 17 is that one gets surprisingly close to $\mathbb{L}_\kappa^{\text{sunf}}$ in $V[G][H]$. One can

handle all B and A, provided that at least one of the them is in $V[G]$. When $B \in V[G]$, then for any $A \in V_\theta[G][H]$ we can ensure $B \in M[G][H^* \cap \mathbb{Q}^{M[G]}]$ by placing a $\mathbb{P}$-name $\dot{B}$ into M. When $A \in V[G]$, then for $B \subseteq \kappa$ in $V[G][H]$ we can work with a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{B}$, and then interpret it by $G * H^*$, rather than by $G * H$, and conclude that there can be no condition forcing that $\dot{B}$ has no embedding working with A. The point here is that if we would try to combine the two arguments, and work in this case with an arbitrary $A \in V_\theta[G][H]$, with a name $\dot{A}$ in $V[G]$, then we would need to know that $\dot{A}_{H^*} = \dot{A}_H$ in order to know that the lifted embedding, with $j(\ell^*)(\kappa) = A$ contradicts the fact that $p \in H^*$. In the actual construction above, we have $A \in V[G]$ and so it has a name $\check{A}$ that is not affected by π.

Statement 2 of Theorem 17 of course includes the case when A is an ordinal, and so in $V[G][H]$ we have an ordinal-anticipating Laver diamond function for strong unfoldability.

Corollary 18 *For none of the principles* $\Ldiamond^{wc}_\kappa$, $\Ldiamond^{unf}_\kappa$, $\Ldiamond^{ind}_\kappa$, $\Ldiamond^{sunf}_\kappa$ *does the existence of an ordinal-anticipating Laver function necessarily imply the existence of a full set-anticipating Laver function. Indeed, the ordinal-anticipating Laver functions do not even imply* $\Diamond_\kappa(\mathrm{REG})$.

Proof: Our model shows that there can be an ordinal anticipating Laver function for strong unfoldability, the strongest of the four principles, but no set-anticipating Laver function for weak compactness, the weakest notion. Since we have $\neg\Diamond_\kappa(\mathrm{REG})$ in our model, the ordinal anticipating Laver functions cannot provide $\Diamond_\kappa(\mathrm{REG})$. $\square$

Corollary 19 *If* $\Ldiamond^{sunf}_\kappa$ *holds in* V*, then in* $V[G][H]$ *there is a sequence* $\langle A_\alpha \mid \alpha \in \mathrm{REG} \cap \kappa \rangle$ *such that*

1. *For any* $A \subseteq \kappa$ *in* $V[G]$*, the set* $\{\, \alpha \mid A \cap \alpha = A_\alpha \,\}$ *is stationary in* $V[G][H]$.
2. *For any* $A \subseteq \kappa$ *in* $V[G][H]$*, the set* $\{\, \alpha \mid A \cap \alpha = A_\alpha \,\}$ *meets every club subset of* κ *in* $V[G]$.

Proof: Let ℓ^* be the function of Theorem 17, and let $A_\alpha = \ell^*(\alpha)$, if this is a subset of α. If $A \subseteq \kappa$ is in $V[G]$, then for any club $C \subseteq \kappa$ in $V[G][H]$ there is by part (2) of Theorem 17 a κ-model M containing A and C and an

embedding $j : M \to N$ with $j(\ell^*)(\kappa) = A$. Since $\kappa \in j(C)$ and $j(A) \cap \kappa = A = j(\ell^*)(\kappa)$, it follows by reflection that $A \cap \alpha = A_\alpha$ for many $\alpha \in C \cap \textsc{reg}$. In the case where $A \in V[G][H]$ and $C \in V[G]$, then one argues similarly using part (1) of Theorem 17. $\square$

These observations show that in any argument producing $\neg\Diamond_\kappa(\textsc{reg})$ for a strongly unfoldable cardinal κ, one should expect the kind of circumlocutions that we went through in the Main Theorem with the unlucky case, rather than a pure lifting argument. To see why, suppose that a pure lifting argument were able to succeed, meaning that every ground model embedding $j : M \to N$ lifts directly to $j : M[G][H \cap \mathbb{Q}^M] \to N[j(G)][j(H \cap \mathbb{Q}^M)]$, where $j(G) = G * H * G_{\text{tail}}$. Any desired $B \subseteq \kappa$ in $V[G][H]$ could be included in $M[G][H \cap \mathbb{Q}^M]$ just by ensuring that M was large enough. If ℓ were a ground model Laver function, then we could define $\ell^*(\alpha)$ to be $\ell(\alpha)_{G_{\alpha+1}}$, and anticipate any $A \in V_\theta[G][H]$ by ensuring $j(\ell)(\kappa) = \dot{A}$ is a name for it. Thus, ℓ^* would be a Laver function in $V[G][H]$, and we couldn't have $\neg\Diamond_\kappa(\textsc{reg})$ there, a contradiction. It follows that the argument of the Main Theorem must often be in the unlucky case, as this is the case that gives rise to the non-lifting circumlocutions.

Lastly, we close by mentioning that $\Diamond_\kappa(\textsc{reg})$ can be forced, while preserving the strong unfoldability of κ.

Theorem 20 ([Ham, Theorem 35]) *If κ is strongly unfoldable, then this is preserved to a forcing extension satisfying $\LaTeX^{sunf}_\kappa$, which therefore also satisfies $\Diamond_\kappa(\textsc{reg})$.*

Proof: We quickly sketch the argument. The results of [Ham01] show that the strong unfoldability of κ is preserved by fast function forcing $\mathbb{F}$, which adds a function $f : \kappa \to \kappa$ such that every $(\theta+1)$-strong unfoldability embedding $j : M \to N$ can be lifted to a $(\theta+1)$-strong unfoldability embedding $j : M[f] \to N[j(f)]$ in $V[f]$, and furthermore for any $\alpha < j(\kappa)$ such a lift can be found with $j(f)(\kappa) = \alpha$. Let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a fixed enumeration of V_κ, and let $\ell(\gamma) = (a_{f(\gamma)})_{f \restriction \gamma}$, provided that $a_{f(\gamma)}$ is an $\mathbb{F}_\gamma$-name, where $\mathbb{F}_\gamma$ is the fast function forcing at γ. This defines the Laver function $\ell : \kappa \to V_\kappa$. Fix any $A \in V_\theta$, and select a name $\dot{A}$ so that $A = \dot{A}_f$ and $\dot{A} \in V_\theta$. If $j : M \to N$ is a ground model $(\theta+1)$-strong unfoldability embedding, with $\vec{a} \in M$, then $\dot{A} = j(\vec{a})(\alpha)$ for some α. We may lift the embedding to $j : M[f] \to N[j(f)]$ with $j(f)(\kappa) = \alpha$. It follows that $j(\ell)(\kappa) = (j(\vec{a})(\alpha))_{j(f)\restriction\kappa} = \dot{A}_f = A$, as desired. So $\LaTeX^{\text{sunf}}_\kappa$ holds in $V[f]$. This easily implies $\Diamond_\kappa(\textsc{reg})$ by simply restricting

ℓ to the values α for which $\ell(\alpha) \subseteq \alpha$. For any $A \subseteq \kappa$ and any club $C \subseteq \kappa$, there is a κ-model M with $A, C, \ell \in M$ and $j : M \to N$ with $j(\ell)(\kappa) = A$. Since $\kappa \in j(C) \cap \textsc{inacc}$, it follows by reflection that $\ell(\alpha) = A \cap \alpha$ for some $\alpha \in C \cap \textsc{inacc}$. So $\diamondsuit_\kappa(\textsc{reg})$ holds. $\square$

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